Variational Calculation on A=3 and 4 Nuclei with Non-Local Potentials

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The application of the hyperspherical harmonic approach to the case of non-local two-body potentials is described. Given the properties of the hyperspherical harmonic functions, there are no difficulties in considering the approach in both coordinate and momentum space. The binding energies and other ground state properties of $A = 3$ and 4 nuclei are calculated using the CD Bonn 2000 and N3LO two-body potentials. The results are shown to be in excellent agreement with corresponding ones obtained by other accurate techniques.

I. INTRODUCTION

A number of "realistic" nucleon-nucleon (NN) potentials have been determined in recent years [1, 2, 3], that reproduce the deuteron binding energy and fit a large set of NN scattering data below the pion-production threshold with a χ^2/datum $\simeq 1$. The high accuracy achieved in reproducing NN observables has spurred renewed interest in testing these potential models in systems with $A \geq 3$. Several methods have been developed to accurately solve the Schrödinger equation for these systems, and meaningful comparisons with precise experimental data, for example for the $N-d$ reaction, are nowadays possible [4, 5]. Among these methods, of particular importance are the Faddeev-Yakubovsky (FY) equations approach [6, 7, 8, 9, 10, 11], the quantum Monte Carlo methods [12, 13], and the techniques based on the expansion of the nuclear wave function on an appropriate basis, like the hyperspherical-harmonics (HH) [14, 15, 16, 17, 18], the harmonic oscillator [19, 20], or the gaussian basis [21, 22]. Some of these methods are variational while others are not, and they are more or less advantageous depending on the problem at hand. All of them provide very accurate results for bound states [23]. On the other hand, the application to scattering states presents a number of difficulties that are specific to each of these methods. Of course, scattering wave functions are an essential input to the calculation of scattering observables as well as of cross sections for electroweak reactions, such as, for example, the low energy radiative and weak capture processes relevant in nuclear astrophysics. The FY approach has been developed both in coordinate and momentum space and, in general, gives very accurate results. However, this approach becomes problematic when considering scattering processes with charged particles at low energies, although recent improvements have been made in Ref. [11] to reduce some of these problems. In this respect, variational methods do not encounter any difficulty, since the electromagnetic interaction between the particles can be taken into account in configuration space [24]. For example, with the HH technique $p-d$ elastic scattering observables have been calculated with the same degree of accuracy as corresponding $n-d$ ones [25, 26, 27].

The failure of theory based the "realistic" NN models in predicting successfully A=3 and 4 bound and scattering state properties provides evidence for the need of including a three nucleon interaction (TNI) [28, 29]. However, the current understanding of the TNI is still in an early stage, and discrepancies between theory and experiments are still observed, as in the case of the $N-d$ [29, 30, 31, 32] and $p-{}^3$He [33, 34] A_y polarization observable (the so-called "A_y-puzzle") It is not clear if these discrepancies can be solved by using more sophisticated models of TNIs, or if the problem still resides in the NN interaction (in particular, in P waves) [35, 36, 37].

More recently, new potentials have been derived using chiral perturbation theory at increasing order [38, 39]. Also many-body forces can be derived on the same footing [40, 41, 42, 43, 44]. Furthermore, there are other models which are constructed up to a certain cutoff momentum Λ (*low-q* potentials [45, 46]). All these new potentials are defined primarily in momentum space and are non-local. Therefore, accurate techniques which can solve the corresponding Schrödinger equation for A=3 and 4 are important. At present, only the FY [8, 10] and the No Core Shell Model [20] methods have been employed for this task. In this paper, we will show that the HH method too can be successfully applied to treat this kind of potentials.

In the last few years, considerable effort has been devoted by the authors of the present paper to the development and application of the HH technique to study bound and scattering states of three or four nucleons with realistic local NN potentials [14, 15, 16, 47, 48, 49]. A version of the method, that has been rather extensively exploited in these calculations, includes an appropriate correlation factor in the HH basis functions, so as to take into account the strong short-range repulsion of the NN interaction and, therefore, improve the rate of convergence of the expansion. This approach is known as the correlated-hyperspherical-harmonics (CHH) method [14, 15, 47]. Up until now, the calculations have been done only for local nuclear interactions, in particular the Argonne v_{18} two-nucleon (AV18) [1] together with the Urbana IX (UIX) three-nucleon [12] interaction.

The HH method, without correlation factors, has been also employed for the calculation of trinucleon [16] and α particle [17] bound states and four-nucleon scattering processes [50]. Such a method appears especially convenient in the case of non-local potentials. The object of the present paper is the calculation of the trinucleon and α particle bound states with the uncorrelated HH expansion using two recent, realistic non-local two-nucleon interactions, namely the CD Bonn 2000 [3] and the N3LO [39] potentials. This kind of expansion, in fact, can be performed equally well in coordinate or momentum space [51]. It is thus possible to treat in the corresponding space the part of interaction given in coordinate or momentum space. With two-body local potentials the matrix elements of the interactions can be obtained via two dimensional integrals. The non-locality of the interaction merely requires a three-dimensional integration, which can be performed using standard numerical methods. Obviously, there are no difficulties in including the local Coulomb interaction. The same is valid for the most commonly used TNIs, which are local interactions. However, the TNI has not been considered in the calculations reported here. As a final remark, it should be noted that the present work is only the "starting point" for the implementation of the HH method in the study of A=3 and 4 scattering states with non-local interactions.

The paper is organized as follows. The details of the formalism are presented in the next section. In Sec. III, the results for the binding energies and ground state properties of systems with A=3 and 4 are presented and compared with those obtained with other different techniques. A few final remarks and conclusions are given in Sec. IV.

II. FORMALISM

In general, an A-body bound state can be written as

$$|\Psi(1,2,\cdots,A)\rangle = \sum_{\mu} c_\mu \, |\Psi_\mu\rangle \ , \tag{2.1}$$

where $|\Psi_\mu\rangle$ are a suitable complete set of states, and μ is an index denoting the set of quantum numbers necessary to completely determine the basis elements. The coefficients of the expansion can be calculated using the Rayleigh-Ritz variational principle, which states that

$$\langle \delta_c \Psi(1,2,\cdots,A) \, | \, H - E \, | \Psi(1,2,\cdots,A)\rangle = 0 \ , \tag{2.2}$$

where $\delta_c\Psi(1,2,\cdots,A)$ indicates the variation of $\Psi(1,2,\cdots,A)$ for arbitrary infinitesimal changes of the linear coefficients c_μ. The problem of determining c_μ and the energy E is then reduced to a generalized eigenvalue problem,

$$\sum_{\mu'} \langle \Psi_\mu \, | \, H - E \, | \, \Psi_{\mu'} \, \rangle \, c_{\mu'} = 0 \ . \tag{2.3}$$

The main difficulty of the method is to compute the matrix elements of the Hamiltonian H with respect to the basis states $|\Psi_\mu\rangle$. Usually H is given as a sum of terms (kinetic energy, two-body potential, etc.). The calculation of the matrix elements of some parts of H can be more conveniently performed in coordinate space, while for other parts it could be easier to work in momentum space. Therefore, it is important that the basis states $|\Psi_\mu\rangle$ have simple expressions in both spaces. The HH functions do have such a property, as will be shown below.

Let us first consider the expression of the HH functions in coordinate space. The internal dynamics of a system of A identical nucleons of mass m is conveniently described in terms of a set of $N = A-1$ Jacobi vectors $\boldsymbol{x}_{1p}, \ldots, \boldsymbol{x}_{Np}$, constructed from a given particle permutation denoted with p, which specifies the particle order $i, j, k, \ldots$. In particular, $\boldsymbol{x}_{Np} = (\boldsymbol{r}_j - \boldsymbol{r}_i)/\sqrt{2}$ ($\boldsymbol{r}_i$ is the position of the i-th particle, etc.), and $p = 1$ is chosen to correspond to the particle order $1, 2, 3, \ldots$. The hyperradial coordinates are defined to be

$$\rho \ = \ \sqrt{\sum_{i=1}^{N} \boldsymbol{x}_{ip}^2} \ , \tag{2.4}$$

$$\Omega_p^{(\rho)} \ = \ [\hat{\boldsymbol{x}}_{1p}, \hat{\boldsymbol{x}}_{2p}, \cdots, \hat{\boldsymbol{x}}_{Np}; \phi_{2p}, \cdots, \phi_{Np}] \ , \tag{2.5}$$

where ρ is the so-called hyperradius, $\Omega_p^{(\rho)}$ a set of angular-hyperangular coordinates and the suffix (ρ) recalls the use of the coordinate space. Note that ρ does not depend on the particle permutation used to construct the Jacobi vectors. The angles $\phi_{2p}, \cdots, \phi_{Np}$ are the hyperangles, defined as [52]

$$\tan \phi_{ip} = \frac{1}{x_{ip}} \sqrt{\sum_{j=1}^{i-1} x_{jp}^2} \ , \qquad i = 2, \ldots, N \ . \tag{2.6}$$

In terms of these variables, chosen any particle permutation p, the kinetic energy operator becomes

$$T = -\frac{\hbar^2}{2m}\sum_{i=1,N}\nabla^2_{\boldsymbol{x}_{ip}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{3N-1}{\rho}\frac{\partial}{\partial\rho} - \frac{\Lambda^2(\Omega_p^{(\rho)})}{\rho^2}\right) . \tag{2.7}$$

The HH functions $Y_{[G]}(\Omega_p^{(\rho)})$ are the eigenfunctions of the operator $\Lambda^2(\Omega_p^{(\rho)})$, and their explicit expression for a generic A-nucleon system can be found, for example, in Refs. [51, 52]. Here we consider in some details only the A=3 case. The Jacobi coordinates for A=3 nucleons are defined as

$$\begin{aligned}\boldsymbol{x}_{2p} &= \frac{1}{\sqrt{2}}(\boldsymbol{r}_j - \boldsymbol{r}_i) ,\\ \boldsymbol{x}_{1p} &= \sqrt{\frac{2}{3}}(\boldsymbol{r}_k - \frac{1}{2}(\boldsymbol{r}_i + \boldsymbol{r}_j)) ,\end{aligned} \tag{2.8}$$

where $p = 1$ corresponds to the order 1,2,3. The HH function $Y^{LL_z}_{[G]}(\Omega_p^{(\rho)})$, with a definite value of the total orbital angular momentum L, L_z, can be written as [16]:

$$Y^{LL_z}_{[G]}(\Omega_p^{(\rho)}) = \left[Y_{\ell_2}(\hat{\boldsymbol{x}}_{2p})\otimes Y_{\ell_1}(\hat{\boldsymbol{x}}_{1p})\right]_{LL_z} N_{[G]}\,(\cos\phi_p)^{\ell_2}(\sin\phi_p)^{\ell_1}\,P^{\ell_1+\frac{1}{2},\ell_2+\frac{1}{2}}_{n_2}(\cos 2\phi_p) . \tag{2.9}$$

Note that in the $A = 3$ system only the hyperangle ϕ_{2p} is present (indicated with ϕ_p). In Eq. (2.9), $Y_{\ell_1}(\hat{\boldsymbol{x}}_{1p})$ and $Y_{\ell_2}(\hat{\boldsymbol{x}}_{2p})$ are spherical harmonics on the two internal Jacobi coordinates $\boldsymbol{x}_{1p}$ and $\boldsymbol{x}_{2p}$, $N_{[G]}$ is a normalization factor and $P^{l_1+\frac{1}{2},l_2+\frac{1}{2}}_{n_2}(\cos 2\phi_p)$ is a Jacobi polynomial, n_2 being the degree of the polynomial. The grand angular quantum number G is defined as $G = 2n_2 + \ell_1 + \ell_2$. The notation $[G]$ stands for $[\ell_1, \ell_2; n_2]$.

Moreover, we consider the anti-symmetrized functions $\mathcal{Y}_{\{G\}}(\Omega^{(\rho)})$ given by the product of an HH and a spin-isospin function. The antisymmetry is obtained by writing $\mathcal{Y}_{\{G\}}(\Omega^{(\rho)})$ as a sum of terms constructed by starting from all the possible even permutations p of the particles, assuming antisymmetry in the interacting pair. For $A = 3$ they are explicitly given by

$$\mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) = \sum_p^{even}\left[Y^{LL_z}_{[G]}(\Omega_p^{(\rho)})\otimes[S_2\otimes\frac{1}{2}]_{SS_z}\right]_{JJ_z}[T_2\otimes\frac{1}{2}]_{TT_z} , \tag{2.10}$$

where the spins (isospins) of particle i and j are coupled to S_2 (T_2), which is itself coupled to the spin (isospin) of the third particle to give the state with total spin S (isospin T, T_z). The total orbital angular momentum L and the total spin S are coupled to the total angular momentum J, J_z. Each set of quantum numbers $\{\ell_1, \ell_2, L, S_2, T_2, S, T\}$ is called "channel", and here the notation $\{G\}$ stands for

$$\{G\} \equiv \{\ell_1, \ell_2, L, S_2, T_2, S, T; n_2\} . \tag{2.11}$$

The requirement of the antisymmetry of $\mathcal{Y}_{\{G\}}(\Omega^{(\rho)})$ constraints the possible choices of the set $\{G\}$ to those for which

$$(-)^{\ell_2+S_2+T_2} = -1 . \tag{2.12}$$

The corresponding expressions for the $A = 4$ system can be found in Ref. [17].

In this paper, we will consider modern two-body potential models which act on specific spin and angular momentum states of the two-body system. Due to the presence of the sum over the permutations in the expression for $\mathcal{Y}_{\{G\}}$, a given particle pair is not in a definite angular and spin state. However, the HH functions with the grand angular quantum number G constructed in terms of a given set of Jacobi vectors $\boldsymbol{x}_{1p}, \cdots, \boldsymbol{x}_{Np}$, defined starting from the order $i, j, k, \ldots$ of particles, can be always expressed in terms of the HH functions constructed, for instance, in terms of $\boldsymbol{x}_{1(p=1)}, \cdots, \boldsymbol{x}_{N(p=1)}$. For example, when $A = 3$, the following relation holds

$$Y^{LL_z}_{[\ell_1,\ell_2;n_2]}(\Omega_p^{(\rho)}) = \sum_{\ell_1',\ell_2',n_2'} a^{(p),L}_{\ell_1,\ell_2,n_2;\,\ell_1',\ell_2',n_2'}\,Y^{LL_z}_{[\ell_1',\ell_2';n_2']}(\Omega^{(\rho)}_{(p=1)}) , \tag{2.13}$$

where the HH functions are defined in Eq. (2.9) and the sum is restricted to the values ℓ_1', ℓ_2', and n_2' such that $\ell_1' + \ell_2' + 2n_2' = G$. The coefficients $a^{(p)}_{\ell_1,\ell_2,n_2;\,\ell_1',\ell_2',n_2'}$ relating the two sets of HH functions are known as the Raynal-Revai coefficients [53]. There exist several procedures to compute these coefficients for $A = 3$ and 4 systems [53, 54,

55, 56, 57, 58]. For $A = 3$, they can be computed rather easily using the orthonormality property of the HH functions, namely

$$a^{(p),L}_{\ell_1,\ell_2,n_2;\,\ell_1',\ell_2',n_2'} = \int d\Omega^{(\rho)}_{(p=1)} \left(Y^{LL_z}_{[\ell_1,\ell_2;n_2]}(\Omega^{(\rho)}_{(p=1)})\right)^* Y^{LL_z}_{[\ell_1',\ell_2';n_2']}(\Omega^{(\rho)}_p) \ . \tag{2.14}$$

For $A = 4$, we have used the procedure devised in Ref. [58], where the corresponding coefficients are obtained by using a set of recurrence relations. Also the spin-isospin states can be recoupled to obtain states where the spin/isospin are coupled in a given order of the particles. The result is that the antisymmetric functions $\mathcal{Y}_{\{G\}}$ can be expressed as a superposition of functions constructed in terms of a given order of particles $i, j, k, \ldots$, each one having the pair i,j in a definite spin and angular momentum state. When the two-body potential acts on the pair of particles i,j, the effect of the projection is easily taken into account.

We now consider the expansion states $|\Psi_\mu\rangle$ of Eq. (2.1). In coordinate space, they have been chosen to be given by

$$\langle\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \,|\, \Psi_\mu \,\rangle = f_l(\rho)\mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \ , \tag{2.15}$$

where μ stands for the set of quantum numbers $(\{G\}, l)$, and $f_l(\rho)$ for $l = 1, \ldots$ is a complete set of hyperradial functions. Note that the index p has been suppressed, since $|\Psi_\mu\rangle$ is at this stage independent on the permutation. The corresponding states in momentum space can be obtained as follows. Let $\hbar\boldsymbol{k}_{1p}, \cdots, \hbar\boldsymbol{k}_{Np}$ be the conjugate Jacobi momenta of the Jacobi vectors. Explicitly, in the A=3 case, these momenta are

$$\begin{aligned} \hbar\boldsymbol{k}_{2p} &= \frac{1}{\sqrt{2}}(\boldsymbol{p}_j - \boldsymbol{p}_i) \ , \\ \hbar\boldsymbol{k}_{1p} &= \sqrt{\frac{2}{3}}(\boldsymbol{p}_k - \frac{1}{2}(\boldsymbol{p}_i + \boldsymbol{p}_j)) \ , \end{aligned} \tag{2.16}$$

$\boldsymbol{p}_i$ being the momentum of the i-th particle. The following relation then may be shown to hold [51, 52]:

$$\begin{aligned} \langle\, \boldsymbol{x}_{1p}, \ldots, \boldsymbol{x}_{Np} \,|\, \boldsymbol{k}_{1p}, \ldots, \boldsymbol{k}_{Np} \,\rangle = & \\ = \ & \frac{1}{(2\pi)^{3N/2}} \exp\!\left(i \sum_{j=1}^{N} \boldsymbol{k}_{jp} \cdot \boldsymbol{x}_{jp}\right) = \\ = \ & \frac{1}{(Q\rho)^{3N/2-1}} \sum_{\{G\}} i^G \, Y_{[G]}(\Omega^{(\rho)}_p) Y^*_{[G]}(\Omega^{(Q)}_p) J_{\mathcal{L}+1/2}(Q\rho) \ , \end{aligned} \tag{2.17}$$

where $\mathcal{L} = G + (3N-3)/2$, $J_{\mathcal{L}+1/2}(Q\rho)$ is a Bessel function, $\Omega^{(\rho)}_p$ is given in Eq. (2.5), and

$$Q = \sqrt{\sum_{i=1}^{N} \boldsymbol{k}_{ip}^2} \ , \tag{2.18}$$

is the hypermomentum, which plays the role in momentum space corresponding to ρ in coordinate space. Also Q does not depend on the particular permutation used to construct the vectors $\hbar\boldsymbol{k}_{1p}, \cdots, \hbar\boldsymbol{k}_{Np}$. The momentum-space angular-hyperangular variables are defined as

$$\Omega^{(Q)}_p = [\hat{\boldsymbol{k}}_{1p}, \hat{\boldsymbol{k}}_{2p}, \cdots, \hat{\boldsymbol{k}}_{Np}; \varphi_{2p}, \cdots, \varphi_{Np}] \ , \tag{2.19}$$

and

$$\tan\varphi_{ip} = \frac{1}{k_{ip}} \sqrt{\sum_{j=1}^{i-1} k_{jp}^2} \ , \qquad i = 2, \ldots, N \ . \tag{2.20}$$

Then, the momentum space version of the wave function given in Eq. (2.15) is

$$\langle\, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N \,|\, \Psi_\mu \,\rangle = \int d\boldsymbol{x}_{1p} \cdots d\boldsymbol{x}_{Np} \, \frac{\mathrm{e}^{-i\sum_{j=1}^{N} \boldsymbol{k}_{jp}\cdot\boldsymbol{x}_{jp}}}{(2\pi)^{3N/2}} f_l(\rho)\mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \ . \tag{2.21}$$

Note that also $\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N \,|\, \Psi_\mu \,\rangle$ cannot depend on the permutation index p. In fact, using the expansion of Eq. (2.17) and the orthogonality property of the HH functions, it follows that

$$\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N \,|\, \Psi_\mu \,\rangle = g_{G,l}(Q) \mathcal{Y}_{\{G\}}(\Omega^{(Q)}) \;, \tag{2.22}$$

where $\mathcal{Y}_{\{G\}}(\Omega^{(Q)})$ is the same as $\mathcal{Y}_{\{G\}}(\Omega^{(\rho)})$ with $\boldsymbol{x}_{ip} \to \boldsymbol{k}_{ip}$, and

$$g_{G,l}(Q) = (-i)^G \int_0^\infty d\rho \, \frac{\rho^{3N-1}}{(Q\rho)^{3N/2-1}} \, J_{\mathcal{L}+1/2}(Q\rho) \, f_l(\rho) \;. \tag{2.23}$$

In the present work, we have adopted two choices for the functions $f_l(\rho)$, for which the Fourier transform given in Eq. (2.23) can be obtained (almost) analytically. The two choices are illustrated below.

1. Exponential functions of the type

$$f_l(\rho) = \mathrm{e}^{-\alpha_l \rho} \;, \tag{2.24}$$

where $\alpha_1 = \beta$ and $\alpha_l = \beta + \gamma^{l-1}$ for $l = 2, \cdots M$, β and γ being two non-linear parameters, which need to be optimized depending on the used nuclear potential model. As an example, when $A = 3$, $\beta = 0.78$ fm^{-1} and $\gamma = 1.2$ fm^{-1} for the CD Bonn 2000 potential. Other choices of α_l have been made, for instance $\alpha_l = \beta + \gamma(l-1)$ or $\alpha_l = \beta\gamma^{l-1}$, but no significant differences have been found in the final results, once the non-linear parameters β and γ are optimized and convergence on M has been reached. The advantage of using the exponential basis (2.24) is that the corresponding functions $g_{G,l}(Q)$ have an easy expression, and are given by:

$$g_{G,l}(Q) = (-i)^G \, \frac{\Gamma(D+G)}{\alpha_l^D} \, \frac{u^D}{(1-u^2)^{\frac{D}{4}-\frac{1}{2}}} P_{D/2}^{1-G-D/2}(u) \;, \qquad u = \sqrt{\frac{1}{1+Q^2/\alpha_l^2}} \;, \tag{2.25}$$

where $D = 3N$ and P_n^m is an associated Legendre function. For $A = 3$ this expression can be written in terms of analytical functions, namely

$$g_{G,l}(Q) = -\frac{(-i)^G}{Q^{G+4}} \, \frac{\sqrt{5!}}{(2\alpha_l)^3} \, \frac{d^3}{d\alpha_l^3} \left[\frac{(\sqrt{\alpha_l^2+Q^2} - \alpha_l)^{G+2}}{\sqrt{\alpha_l^2+Q^2}} \right] \;. \tag{2.26}$$

2. Another useful form for $f_l(\rho)$, adopted for example also in Ref. [59], is

$$f_l(\rho) = \gamma^{D/2} \sqrt{\frac{l!}{(l+D-1)!}} \; L_l^{(D-1)}(\gamma\rho) \; \mathrm{e}^{-\frac{\gamma}{2}\rho} \;, \tag{2.27}$$

where $L_l^{(D-1)}(\gamma\rho)$ are Laguerre polynomials. Here, there is only one non-linear parameter, γ, to be variationally optimized. In particular, γ can be chosen in the interval 3.5–4.5 fm^{-1} for the CD Bonn 2000 potential and 6–8 fm^{-1} for the N3LO potential, for both $A = 3$ and 4. The corresponding functions $g_{G,l}(Q)$ are less trivial to calculate, and are given by:

$$g_{G,l}(Q) = \frac{(-i)^G}{\gamma^{D/2}} \sqrt{\frac{l!}{(l+D-1)!}} \sum_{k=0}^{l} b_k^l \, 2^{k+D} \; \Gamma(G+k+D) \; \frac{u^{k+D}}{(1-u^2)^{\frac{D}{4}-\frac{1}{2}}} \; P_{k+D/2}^{1-G-D/2}(u) \;, \tag{2.28}$$

where $u = \frac{1}{\sqrt{1+(2Q/\gamma)^2}}$ and b_k^l are given by

$$b_k^l = \frac{(-1)^k}{k!} \begin{pmatrix} l+D-1 \\ l-k \end{pmatrix} \;, \tag{2.29}$$

so that $L_l^{(D-1)}(x) = \sum_{k=0}^{l} b_k^l \, x^k$ [60]. Studies on the convergence on M for the A=3 and 4 calculations are presented in the Sec. III.

In summary, the variational state is given by

$$|\, \Psi(1, \ldots, A) \,\rangle = \sum_{\{G\}} \sum_{l=1}^{M} c_{\{G\},l} |\, \Psi_{\{G\},l} \,\rangle \;, \tag{2.30}$$

where, in momentum space the expansion states are

$$\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_N \,|\, \Psi_{\{G\},l} \rangle = g_{G,l}(Q) \mathcal{Y}_{\{G\}}(\Omega^{(Q)}) \ , \tag{2.31}$$

and in coordinate space are

$$\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \,|\, \Psi_{\{G\},l} \rangle = f_l(\rho) \mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \ . \tag{2.32}$$

These two expressions can be used to evaluate the matrix elements in Eq. (2.3). In particular, the normalization (N) and kinetic energy (T) operator matrix elements can be computed both in coordinate and in momentum space. Explicitly:

$$\begin{aligned} N_{\{G'\},l';\{G\},l} &= \int d\rho\, \rho^{3N-1} f_{l'}(\rho) f_l(\rho) \int d\Omega^{(\rho)}\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(\rho)})\, \mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \\ &= \int dQ\, Q^{3N-1} g_{G',l'}(Q)\, g_{G,l}(Q) \int d\Omega^{(Q)}\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(Q)})\, \mathcal{Y}_{\{G\}}(\Omega^{(Q)}) \ , \end{aligned} \tag{2.33}$$

$$\begin{aligned} T_{\{G'\},l';\{G\},l} &= -\frac{\hbar^2}{2m} \int d\rho\, \rho^{3N-1} f_{l'}(\rho) \left[\frac{\partial^2}{\partial \rho^2} + \frac{3N-1}{\rho} \frac{\partial}{\partial \rho} - \frac{G(G+3N-2)}{\rho^2} \right] f_l(\rho) \\ &\quad \times \int d\Omega^{(\rho)}\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(\rho)})\, \mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \\ &= \frac{\hbar^2}{2m} \int dQ\, Q^{3N+1} g_{G',l'}(Q)\, g_{G,l}(Q) \\ &\quad \times \int d\Omega^{(Q)}\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(Q)})\, \mathcal{Y}_{\{G\}}(\Omega^{(Q)}) \ , \end{aligned} \tag{2.34}$$

where it has been used the fact that the HH functions are eigenfunctions of the operator $\Lambda^2(\Omega_p^{(\rho)})$ defined in Eq. (2.7) corresponding to the eigenvalues $G(G+3N-2)$. In Eqs. (2.33) and (2.34), as well as in the rest of the present work, the permutation index p is omitted, and the integration variables corresponding to $p=1$ are used (i.e. $d\Omega^{(\rho/Q)} \equiv d\Omega^{(\rho/Q)}_{(p=1)}$).

The calculation of the two-body potential energy matrix elements is more conveniently performed in either coordinate or momentum space, depending on the particular potential model of interest. First of all, due to the antisymmetry of the wave function, the following relation holds

$$V_{\{G'\},l';\{G\},l} \equiv \langle \Psi_{\{G'\},l'} | V | \Psi_{\{G\},l} \rangle = \frac{A(A-1)}{2} \langle \Psi_{\{G'\},l'} | v(1,2) | \Psi_{\{G\},l} \rangle \ , \tag{2.35}$$

where $v(1,2)$ acts on the particle pair 1,2 and can be written as

$$v(1,2) = V(\boldsymbol{x}'_N; \boldsymbol{x}_N) \ , \tag{2.36}$$

in coordinate space, and

$$v(1,2) = \widetilde{V}(\boldsymbol{k}'_N; \boldsymbol{k}_N) \ , \tag{2.37}$$

in momentum space (clearly $\widetilde{V}$ and V are related by a Fourier transform). Then,

$$\begin{aligned} V_{\{G'\},l';\{G\},l} &= \frac{A(A-1)}{2} \int d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_N \int d\boldsymbol{x}'_N\ f_{l'}(\rho')\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(\rho')}) \\ &\quad \times V(\boldsymbol{x}'_N; \boldsymbol{x}_N) f_l(\rho)\, \mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} V_{\{G'\},l';\{G\},l} &= \frac{A(A-1)}{2} \int d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N \int d\boldsymbol{k}'_N\ g_{G',l'}(Q')\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(Q')}) \\ &\quad \times \widetilde{V}(\boldsymbol{k}'_N; \boldsymbol{k}_N) g_{G,l}(Q)\, \mathcal{Y}_{\{G\}}(\Omega^{(Q)}) \ , \end{aligned} \tag{2.39}$$

where $\rho', \Omega^{(\rho')}$ are the hyperradial coordinates associated to the Jacobi vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{(N-1)}, \boldsymbol{x}'_N$, etc. When the potential energy operator is a local operator in coordinate space, namely

$$V(\boldsymbol{x}'_N; \boldsymbol{x}_N) \to V_{loc}(\boldsymbol{x}_N) \delta(\boldsymbol{x}'_N - \boldsymbol{x}_N) \ , \tag{2.40}$$

it is more convenient to calculate the corresponding matrix elements using Eq.(2.38), which simplifies to

$$V_{\{G'\},l';\{G\},l} = \int d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_N f_{l'}(\rho)\, \mathcal{Y}^*_{\{G'\}}(\Omega^{(\rho)})\, V_{loc}(\boldsymbol{x}_N) f_l(\rho)\, \mathcal{Y}_{\{G\}}(\Omega^{(\rho)}) \ . \tag{2.41}$$

Examples of two-nucleon potential models of this form are the Argonne v_{18} [1] or the Nijmegen II [2] potentials. Note that previous applications of the HH method were limited to these cases in Refs. [16, 17]. On the other hand, for the CD Bonn 2000 or N3LO potentials, which are non-local operators in momentum space, the use of the momentum-space expression of $V_{\{G'\},l';\{G\},l}$ is more convenient. In general, the potential energy operator is given as a sum of different local and/or non-local terms. The computation of the matrix elements of each part can be performed using either the coordinate- or the momentum-space expression, depending on the convenience. For instance, if the selected potential energy model includes the CD Bonn 2000 potential and the (point) Coulomb interaction, the matrix elements of the CD Bonn 2000 are computed using Eq. (2.39), the ones of the Coulomb potential are computed using Eq. (2.41). The same procedure can be applied to the TNIs mostly used in the literature, i.e. the Urbana-type [12] and Tucson-Melbourne-type [61, 62] potentials. However, since the aim of this work is to study the applicability of the HH method when non-local potentials are used, the TNI has not been included.

Other potentials models which are frequently used in the literature are those developed by Doleschall *et al.* [63, 64]. These models consist of non-local operators given in coordinate space. In that case, it is more convenient to perform the calculation using Eq. (2.38).

The calculation of the integrals involved in Eqs. (2.38) or (2.39) is a non-trivial numerical task. As an example, let us consider Eq. (2.39). Remembering that $\widetilde{V}(\boldsymbol{k}'_N;\boldsymbol{k}_N)$ acts on the particle pair 1,2, it is convenient, using the Raynal-Revai coefficients and the recoupling of spin-isospin states, to express the states $\langle\Psi_{\{G'\},l'}|$ and $|\Psi_{\{G\},l}\rangle$ in momentum space as a superposition of HH functions and spin-isospin states constructed using the order of particles $1,2,3,\ldots,A$. Most of the integrations can be now performed analytically and the matrix elements, for a general A-body system, reduces to a sum of three-dimensional integrals of the type

$$\begin{aligned} I^{l',l;j,S',S}_{G',\ell'_N,n'_N;G,\ell_N,n_N} \ = \ & \int_0^\infty dq\, q^{D-4} \int_0^\infty dk_N\ (k_N)^2 \int_0^\infty dk'_N (k'_N)^2 g_{G',l'}(Q') \\ & \times (\cos\varphi'_N)^{\ell'_N} (\sin\varphi'_N)^{\nu'} P^{\nu'+D/2-5/2,\ell'_N+1/2}_{n'_N}(\cos 2\varphi'_N) v^j_{\ell'_N,S';\ell_N,S}(k'_N,k_N) \\ & \times g_{G,l}(Q) (\cos\varphi_N)^{\ell_N} (\sin\varphi_N)^{\nu} P^{\nu+D/2-5/2,\ell_N+1/2}_{n_N}(\cos 2\varphi_N) \delta_{\nu,\nu'} \ , \end{aligned} \tag{2.42}$$

where

$$Q^2 = k_N^2 + q^2 \ , \quad (Q')^2 = (k'_N)^2 + q^2 \ , \quad \cos\varphi_N = k_N/Q \ , \quad \cos\varphi'_N = k'_N/Q' \ , \tag{2.43}$$

and

$$\nu = G - 2n_N - \ell_N \ , \qquad \nu' = G' - 2n'_N - \ell'_N \ . \tag{2.44}$$

Moreover, in Eq. (2.42) $v^j_{\ell'_N,S';\ell_N,S}(k'_N,k_N)$ is the two-body potential acting between states of the pair of particles 1, 2 of total angular momentum j, and orbital angular momentum and spin quantum numbers ℓ'_N, S' (on the left) and ℓ_N, S (on the right). The integrals $I^{l',l;j,S',S}_{G',\ell'_N,n'_N;G,\ell_N,n_N}$ can be computed beforehand and stored in computer disks. The last step consists in combing $I^{l',l;j,S',S}_{G',\ell'_N,n'_N;G,\ell_N,n_N}$ with the Raynal-Revai coefficients to obtain the matrix elements of Eq. (2.39). Finally, the integrations involved in Eq. (2.42) can be accurately performed with standard numerical techniques (Gauss integration) [60].

III. RESULTS

In this section, the binding energy and ground-state properties obtained for the nuclear systems with A=3 and 4 are presented for the CD Bonn 2000 [3] and the N3LO [39] momentum space potentials. These interaction models are decomposed on partial waves, and the partial wave decomposition is truncated at a certain value of the two-body total angular momentum j_{max}. In the present work j_{max}=6 has been chosen, which allows for an accuracy of better than 1 keV in the triton binding energy.

The section is divided into three subsections: in Sec. III A and III B the convergence of the expansion with respect to the quantum numbers $\{G\}$ and the number M of hyperradial functions is discussed for A=3 and 4, respectively. The converged results for the ground-state properties of triton, ^{3}He and ^{4}He are presented in Sec. III C, and compared with the results obtained with different approaches.

TABLE I: Triton binding energies in MeV, calculated with the CD Bonn 2000 two-nucleon interaction, using only the first 3 expansion channels and the exponential functions of Eqs. (2.24) and (2.26) as expansion basis for the hyperradial and hypermomentum functions. M is the maximum number of basis elements, G is the grand angular momentum.

G	M=8	M=10	M=12	M=14	M=16
40	7.6764	7.6835	7.6842	7.6845	7.6846
50	7.6845	7.6921	7.6934	7.6934	7.6935
60	7.6870	7.6948	7.6958	7.6962	7.6963
70	7.6880	7.6958	7.6969	7.6974	7.6976
80	7.6885	7.6964	7.6974	7.6979	7.6981

TABLE II: Same as Table I but with the Laguerre polynomial expansion of Eqs. (2.27) and (2.28) for the hyperradial and hypermomentum functions.

G	M=8	M=10	M=12	M=14	M=16	M=18	M=20	M=22	M=24
40	7.6582	7.6773	7.6827	7.6835	7.6842	7.6846	7.6847	7.6848	7.6848
50	7.6665	7.6862	7.6916	7.6924	7.6931	7.6936	7.6937	7.6937	7.6937
60	7.6691	7.6892	7.6946	7.6955	7.6962	7.6966	7.6967	7.6967	7.6967
70	7.6702	7.6904	7.6959	7.6967	7.6974	7.6978	7.6979	7.6980	7.6980
80	7.6706	7.6910	7.6965	7.6973	7.6980	7.6984	7.6985	7.6986	7.6986

A. Convergence of the A=3 Results

In this subsection, we show the level of accuracy reached by the method presented in this work. As an example, in Table I and II, the triton binding energy is calculated with the CD Bonn 2000 potential using only the first 3 channels of Table I of Ref. [16], and increasing the value of the grand angular momentum G and the value of the number M of basis elements in the expansion of the hyperradial and hypermomentum functions $f_l(\rho)$ and $g_{G,l}(Q)$ (see Eqs. (2.15) and (2.22)). In Table I the exponential basis of Eqs. (2.24) and (2.26) is used, and the non-linear parameters β and γ are 0.78 fm^{-1} and 1.2 fm^{-1} respectively. In Table II, $f_l(\rho)$ and $g_{G,l}(Q)$ are expanded on Laguerre polynomials, Eqs. (2.27) and (2.28), and the non-linear parameter γ is 4.0 fm^{-1}.

By inspection of the tables, we can conclude that i) the convergence of the exponential basis at the 1 keV accuracy level is slightly faster than the one of the Laguerre polynomials basis. In fact, the maximum value of M used in the two expansions, M_{max}, is 16 and 24 respectively. However, it should be noted that the exponential basis gives numerical problems for higher values of M_{max}, and turns out to be not suitable for the N3LO potential. Nevertheless, the binding energy B at G=80 calculated with $M_{max} = 16$ exponentials is only 0.5 keV smaller than the result obtained with $M_{max} = 24$ Laguerre polynomials. ii) In the case of the exponential basis, the expansion on M is truncated when $B(M_{max}) - B(M_{max} - 2) < 0.5$ keV. In the case of the Laguerre polynomial expansion, $B(M_{max}) - B(M_{max} - 2)$ is smaller than a tenth of keV. iii) The convergence on G is reached at G_{max}=80. In fact, in Table I, we can see that going from G=60 to G=70 and from G=70 to G=80, the gain in B is 1.3 keV and 0.5 keV, respectively. In Table II, the corresponding values are 1.3 keV and 0.6 keV, respectively. Due to these considerations, we are confident in concluding that the results of 7.699 MeV for the triton binding energy, obtained with only the first 3 channels in our channel expansion, is accurate at least at the 1 keV level. Note that the value for G_{max} used in Ref. [16] for the AV18 potential and the first 3 expansion channels is 180. This is related to the fact that both the CD Bonn 2000 and N3LO potential models have a rather soft short-range repulsion compared with the AV18 potential model.

A similar procedure to reach the convergence on G and M has been used also when other channels are included in the calculation. The converged A=3 results presented in Sec. III C have been obtained including up to 23 channels, 18 having total isospin T=1/2 and 5 having T=3/2 (see Sec. II and Ref. [16]). In particular, for the CD Bonn 2000 (N3LO) potential, the maximum value of the grand angular momentum G_{max} is 80 (40) for the first 3 channels, 40 (20) for the next 5 ones, and 30 (20) for the last 10 channels with T=1/2. The convergence for the T=3/2 channels is reached using G_{max}=20.

B. Convergence of the A=4 Results

First of all, we discuss the convergence of the expansion with respect to the quantum numbers $\{G\}$. We follow the procedure adopted in Ref. [17]. The states $|\Psi_{\{G\},l}\rangle$ are separated in six *classes*, depending on the total isospin

TABLE III: Convergence of α–particle binding energy B (MeV), mean value of the kinetic energy $\langle T\rangle$ (MeV), mean square radius $\sqrt{\langle r\rangle^2}$ (fm) and $P-$ and $D-$wave percentages for different values of M, the number of functions $g_{G,l}(Q)$ included in the expansion. In this example, $g_{G,l}(Q)$ are obtained expanding the hyperradial functions on Laguerre polynomials with $\gamma = 6.0$ fm^{-1}. The potential model considered is the N3LO. The basis includes HH states with $\{G_1, G_2, G_3, G_4, G_5, G_6\} = \{24, 20, 16, 16, 16, 20\}$ (see the text for more details).

M	B	$\langle T\rangle$	$\sqrt{\langle r^2\rangle}$	P_P	P_D
8	24.649	72.326	1.453	0.172	9.272
10	25.195	69.940	1.495	0.173	9.315
12	25.339	69.309	1.512	0.172	9.287
14	25.362	69.244	1.514	0.172	9.290
16	25.373	69.230	1.516	0.172	9.289
18	25.376	69.236	1.516	0.172	9.289

quantum number T and on ℓ_1, ℓ_2, ℓ_3, the values of the orbital angular momentum quantum numbers associated to the three Jacobi vectors (or momenta). The most important classes are those ones with $T = 0$ and with the lowest values of $\ell_1+\ell_2+\ell_3$. The convergence of the class i is studied by including in the expansion all the states $\mid \Psi_{\{G\},l} \rangle$ with grand angular quantum number $G \leq G_i$, and then increasing the value of G_i. The rate of convergence depends primarily on the repulsion at short interparticle distance of the adopted potential models. More details on this argument and the choice of the basis can be found in Ref. [17].

In the case of the two potential models considered in this work, it turns out that both the models have a rather soft repulsion and the convergence is reached for values of G_i, $i = 1, \ldots, 6$ rather smaller than those found in Ref. [17], where the the AV18 [1] and Nijmegen II [2] potential models were considered. To give an example, for those models the convergence was reached for $\{G_1, G_2, G_3, G_4, G_5, G_6\} = \{72, 40, 34, 28, 24, 20\}$, while for the CD Bonn 2000 potential a satisfactory convergence is obtained for $\{G_1, G_2, G_3, G_4, G_5, G_6\} = \{52, 34, 28, 24, 20, 20\}$. The N3LO potential is derived from an effective field theory by means of an expansion valid at low momenta, and it is strongly suppressed at high momenta. This corresponds in coordinate space to a rather soft repulsion at short interparticle distances. Correspondingly, a satisfactory convergence is reached at $\{G_1, G_2, G_3, G_4, G_5, G_6\} = \{24, 20, 16, 16, 16, 20\}$. The "missing binding energy" due to the truncation of the basis has been estimated, using the same procedure adopted in Ref. [17], to be around 10 keV for the CD Bonn 2000 case, and around 5 keV for the N3LO potential.

We now consider the convergence with respect to the number M of functions $g_{G,l}(Q)$ used in Eq. (2.30). In Table III, the binding energy and other ground state properties calculated for the N3LO potential for different values of M are reported. In this example, the hyperradial functions have been expanded on Laguerre polynomials with $\gamma = 6.0$ fm^{-1}. All the HH functions belonging to the six classes discussed above, namely for the choice $\{G_1, G_2, G_3, G_4, G_5, G_6\} = \{24, 20, 16, 16, 16, 20\}$, have been included. From the table it is seen that all quantities have converged quite well for $M \approx 16 - 18$.

C. Results for A=3 and 4

The converged values for the triton, ^{3}He and ^{4}He binding energies and other ground-state properties are listed in Table IV, V and VI, respectively. All these results have been obtained expanding the hyperradial functions on Laguerre polynomials, with γ=4 fm^{-1} and 7 fm^{-1} for the CD Bonn 2000 and N3LO potentials, respectively. The numerical uncertainty in the triton and ^{3}He binding energy has been estimated to be at the most of the order of 1 keV, and in the ^{4}He binding energy of the order of 10 keV. These results are compared with those obtained with other approaches. In particular, we have considered the Faddeev (for A=3) and Faddeev-Yakubovsky (FY) (for A=4) approach [10, 65, 66, 67], and the No Core Shell Model (NCSM) approach [20, 68]. For sake of comparison, the results for the AV18 potential are also listed [8, 16, 17, 65, 69]. Note that the HH results of Ref. [16] do not include the $T = 3/2$ states. In the ^{3}He and ^{4}He case, all the results for the CD Bonn 2000 and N3LO potentials have been obtained including the point Coulomb interaction, except for the ^{4}He FY results with the N3LO potential. The latter have been obtained including a more complicated electro-magnetic interaction between nucleons [66], whose effects are however quite similar to those of the point Coulomb interaction.

There is a good agreement, at the level of 0.1 %, among the results obtained with different techniques for all quantities considered in the tables. Furthermore, the mean value of the kinetic energy for the N3LO case is smaller than that found with the AV18 potential. This is due to the fact that the repulsion at short interparticle distances is softer for the N3LO potential than for the AV18 potential. Also the percentages of the P- and D-waves is significantly smaller for the N3LO potential. This explains the faster convergence of the HH expansion in this case. The CD Bonn

TABLE IV: The triton binding energies B (MeV), the mean square radii $\sqrt{\langle r^2 \rangle}$ (fm), the expectation values of the kinetic energy operator $\langle T \rangle$ (MeV), and the mixed-symmetry S', P, D, and isospin T=3/2 probabilities (all in %), calculated with the CD Bonn 2000 and N3LO potentials, are compared with the results obtained within the Faddeev equations approach [10, 65, 66, 67] (FE) and within the No Core Shell Model (NCSM) approach [20, 68]. The results obtained in Refs. [16] and [65] within the HH, correlated-hyperspherical-harmonics (CHH) and FE approaches for the AV18 potential have been also reported for sake of comparison. These last HH results do not include the T=3/2 states.

Interaction	Method	B	$\langle T \rangle$	$\sqrt{\langle r^2 \rangle}$	$P_{S'}$	P_P	P_D	$P_{T=3/2}$
CD Bonn 2000	HH (this work)	7.998	37.630	1.721	1.31	0.047	7.02	0.0049
	FE [65, 66]	7.997	37.620	-	1.31	0.047	7.02	0.0048
	FE [10, 67]	7.998	37.627	-	1.31	0.047	7.02	0.0048
	NCSM [20, 68]	7.99(1)	-	-	-	-	-	-
N3LO	HH (this work)	7.854	34.555	1.758	1.36	0.037	6.31	0.0009
	FE [65, 66]	7.854	34.546	-	1.37	0.037	6.32	0.0009
	FE [10, 67]	7.854	34.547	-	1.37	0.037	6.32	0.0009
	NCSM [20, 68]	7.85(1)	-	-	-	-	-	-
AV18	CHH [65]	7.624	46.727	-	1.293	0.066	8.510	0.0025
	HH [16]	7.618	46.707	1.770	1.294	0.066	8.511	-
	FE [65]	7.621	46.73	-	1.291	0.066	8.510	0.0025

TABLE V: Same as Table IV but for ^{3}He.

Interaction	Method	B	$\langle T \rangle$	$\sqrt{\langle r^2 \rangle}$	$P_{S'}$	P_P	P_D	$P_{T=3/2}$
CD Bonn 2000	HH (this work)	7.262	36.777	1.759	1.54	0.046	7.00	0.0109
	FE [65, 66]	7.261	36.756	-	1.54	0.046	7.00	0.0110
	FE [10, 67]	7.263	36.761	-	1.54	0.046	7.00	0.0110
N3LO	HH (this work)	7.128	33.789	1.797	1.61	0.037	6.31	0.0062
	FE [65, 66]	7.128	33.775	-	1.61	0.037	6.31	0.0063
	FE [10, 67]	7.128	33.775	-	1.61	0.037	6.32	0.0063
AV18	CHH [65]	6.925	45.685	-	1.530	0.065	8.467	0.0080
	FE [65]	6.923	45.68	-	1.524	0.065	8.466	0.0081

2000 is intermediate between the two cases discussed above.

We now consider the results for the T=3/2 and $T > 0$ components in the A=3 and 4 systems, respectively. The percentage of the $T = 1$ component in the ^{4}He ground state wave function is almost independent from the adopted potential model. In fact, as discussed in Ref. [17], 50% of it is due to the effect of the Coulomb interaction between the protons, the remaining 50% is due to the charge symmetry breaking (CSB) terms in the nuclear interaction. Consequently, the difference in $P_{T=1}$ for the various two-body potentials is small. On the contrary, the T=3/2 component in triton and the $T = 2$ component in ^{4}He are largely dominated by CSB of nuclear origin (different pion masses, etc.). The values reported in Table IV and VI show that, depending on the interaction, rather different values for the triton $P_{T=3/2}$ and the ^{4}He $P_{T=2}$ are obtained (note that the standard models of TNI have little effect on the isospin admixtures [17]). The CSB magnitude in the two-body interaction is fixed by fitting the NN scattering data, and therefore comes from differences observed in the pp and np systems. In particular, the CD Bonn 2000 and N3LO potentials fit the same NN data set [70]. The origin of the rather large differences found for the triton $P_{T=3/2}$ and the ^{4}He $P_{T=2}$ (a factor 5 between CD Bonn 2000 and N3LO) must be related to quite different off-shell behavior of the CSB terms of both interactions. Note that the T=3/2 component in ^{3}He is also affected by the Coulomb interaction, which reduces by more than a factor of 2 the difference between the CD Bonn 2000 and N3LO $P_{T=3/2}$ results.

Finally, we observe that knowledge of the $T = 1$ and 2 percentages could be important for parity-violating electron scattering experiments on ^{4}He, aimed at studying admixture of strange quark $s\bar{s}$ pairs in nucleons and nuclei [71, 72, 73]. It could play an important role also in the study of the reaction $d + d \to \alpha + \pi^0$. This reaction is possible only if isospin symmetry is violated, namely it probes directly the CSB terms in the nuclear Hamiltonian [74, 75].

TABLE VI: The α–particle binding energies B (MeV), the mean square radii $\sqrt{\langle r^2 \rangle}$ (fm), the expectation values of the kinetic energy operator $\langle T \rangle$ (MeV), and the P, D, $T = 1$ and $T = 2$ probabilities (%) for the two non-local potentials considered in this paper. The results obtained in Ref. [17] for the AV18 potential have been also reported for sake of comparison. The results obtained by other techniques are also listed.

Interaction	Method	B	$\langle T \rangle$	$\sqrt{\langle r^2 \rangle}$	P_P	P_D	$P_{T=1}$	$P_{T=2}$
CD Bonn 2000	HH (this work)	26.13	77.58	1.454	0.223	10.74	0.0029	0.0108
	FY [66]	26.16	77.59	-	0.225	10.77	0.0030	0.0108
N3LO	HH (this work)	25.38	69.24	1.516	0.172	9.289	0.0035	0.0024
	FY [66]	25.37	69.20	-	0.172	9.293	0.0033	0.0024
	NCSM [20]	25.36(4)	-	-	-	-	-	-
AV18	HH [17]	24.210	97.84	1.512	0.347	13.74	0.0028	0.0052
	FY [8]	24.25	97.80	-	0.35	13.78	0.003	0.005
	FY [69]	24.223	97.77	1.516	-	-	-	-

IV. SUMMARY AND CONCLUSIONS

The $A = 3$ and 4 nuclear ground states have been studied with non-local two-body potentials using the HH method. The variational wave function is written as an expansion over a complete basis, which is constructed in coordinate space as a product of hyperradial, HH and spin-isospin functions. The main task is the calculation of the Fourier transform of the expansion basis. However, given the properties of the HH functions, this Fourier transform reduces to a one-dimensional integral which can be obtained analytically. The application is therefore rather straightforward, and the matrix elements of a given two-body interaction can always be reduced to three-dimensional integrals (two-dimensional for local potentials). We have presented the results for the CD Bonn 2000 and N3LO NN potential models. In both cases we have found very good agreement with the results obtained by other groups. We have also given the estimates obtained for various ground state properties and pointed out that the different models predict very different (up to a factor 5) percentages of the isospin admixtures in the ^{3}H and ^{4}He ground states.

In recent years, new models of nuclear interaction have been constructed from chiral perturbation theory. They are non-local and given in momentum space [38]. The N3LO potential is the first potential of this type fitting the NN data set with χ^2/datum $\simeq 1$. Another class of recently developed potentials are the so-called *low-q* potentials [45, 46]. They are "renormalized" two-body potentials $\widetilde{V}_{NN}(k, k')$, given in momentum space, where the tail for high values of k and k' has been eliminated.

It is important, therefore, to have accurate techniques to solve the Schrödinger equation for non-local momentum-space potentials. Up to now, only the FY [8, 10] and the NCSM [20] methods were available for this task. In this paper, we have shown that also the HH method can be successfully applied to treat this kind of potentials.

There are two other important motivations behind this work. The first one is that the HH formalism can be applied also to scattering problems. As discussed in Sec. I, there are still a number of theoretical problems in the $N - d$ and $p - {}^3$He reactions, and it would be very interesting to check whether the use of these new potential models could solve these problems. In particular, it appears promising the development of more realistic models of TNIs to be tested in $A = 3$ and 4 nuclear systems. The second motivation is the possibility of the extending the HH method to larger systems. The feasibility of such an application would require the solution of several problems, having to do with the fast computation of the Raynal-Revai coefficients for $A > 4$ and the very large degeneracy of the basis. However, since these new potentials are softer than the older ones, it should be not too difficult to solve this latter problem. Work in both directions is currently underway.

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